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THE MODERN OPTION PRICING THEORY: A REVIEW

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1. Introduction

Options, as well as any other derivative contracts, are developed to trade abstract claims such as rights or obligations. Besides, an option value is directly related to the future movement of its underlying asset price, which is not evident and have to be forecasted. Measuring the optimal price of these contracts, therefore, is much more difficult than pricing tangible commodities that are traded in physical markets. Although the path-breaking work of Black and Scholes (1973, BS hereafter) establish the fundamental basis of modern option pricing theory, there has been a controversy over the appropriateness of proposed theories and models. Therefore, it is necessary to know which topics are under discussion and what kind of solutions are proposed, to properly understand the history of modern option pricing theory.

BS prove that variables such as the expected rate of return of the underlying asset, or the investors attitude to risk do not appear in the option pricing equation.

They show that only five variables are needed to price an option contract: underlying asset price, variance of the underlying asset price, strike price of the contract, time to maturity of the contract, and risk-free interest rate. Except the variance of the underlying, the other four variables can be easily obtained from the market. If we correctly measure the variance, therefore, we can find how much should be paid for an option contract.

However, BS also have some deficiencies; especially, it is found that the assumption of a continuous time stock price process with a constant variance rate is not reliable. Many evidences, such as sudden crashes in stock markets or the anomaly of 'volatility smile', show that the assumption is not realistic and the theoretical price under the assumption would be biased in the real markets. The sudden crashes reveal that the stock price movement is sometimes not continuous; the volatility smile shows that the variance rate is not constant. Moreover, these anomalies became more difficult to be explained with the original BS framework after the volatility smile changed its shape after the stock market crash of 1987.

Academics and practitioners on derivative securities have been trying to find why these anomalies exist, and what is the best way to cope with them when pricing options. Because the price process assumption has its deficiencies in many aspects, various theories and models are developed to deal with some specific shortfalls of the assumption. Jump models introduce a separated Poisson jump process to explain sudden steep changes; stochastic volatility models release the assumption of constant variance rate to explain the volatility smile; Net buying pressure theory argue that the supply and demand curves also affect the volatility smile. Each model and theory has its strength since all of them succeed to solve a part of the problem regarding the price process assumption.

This paper tries to explain the goal of these theories and models, and examines how they deal with the problems in the price process assumption. First, we examine the parametric approaches. We try to describe how the studies set their models and parameters to assume the underlying asset price dynamics, and explain how the models introduce the concepts of jumps and stochastic volatility into the process. Parameter estimation methods are also examined to understand how previous studies actually employ their models. Next, we broaden our focus to the non-parametric approaches. Non-parametric approaches enable option pricing in a model-free manner. If we set a model to estimate the asset price dynamics, model parameters are calibrated to satisfy certain conditions. This way of estimation may cause a model risk, which comes from the flaws in model assumptions. On the other hand, non-parametric approaches try to avoid the model risk by assuming no specific model. Finally, we take a quick view on the net buying pressure theory which argues that the supply and demand curves affect option prices. The theory suggests that the volatility smile is caused by the imperfections in market microstructure, not the stochastic volatility.

The outline of this paper is as follows. Section 2 reviews BS to understand the fundamental concepts. Section 3 examines the modern parametric option pricing models and parameter estimation methods. Section 4 describes the attributes of the non-parametric approaches. Section 5 introduces the net buying pressure theory. Section 6 concludes this paper.

2. The Black and Scholes (1973) framework

2.1. **Risk-neutral option pricing.** There are a number of studies which try to define general option pricing equations before BS. Some of their pricing equations are very similar to the Black-Scholes equation. For example, Sprenkle (1961) defines the formula of the value of a European call option as follows:

$$c = \alpha S N(d_1) - \beta K N(d_2) \ , \tag{2.1}$$

where

$$d_1 = \frac{\ln(\alpha S/K) + \sigma^2 T/2}{\sigma\sqrt{T}} \ , d_2 = \frac{\ln(\alpha S/K) - \sigma^2 T/2}{\sigma\sqrt{T}} \ .$$

In this expression, S is the stock price at present, K is the exercise price, T is the time to maturity of the option (in years), σ^2 is the variance rate of the return on the stock, and $N(x)$ is the cumulative standard normal density function. Although all of the four variables above, S, K, T, and σ^2 can be observed in the markets, α and β are unknown. Sprenkle (1961) defines α as the expected rate of return of the stock at the maturity, and β as a discount factor that depends on the risk of the stock. Parameters α and β are needed to consider the time value of money. In finance, a dollar now in hand has more value than a dollar tomorrow because of the chance of investment. Therefore, we need to discount future payoffs with proper discount factors to derive their present value. He shows that, however, it is unable to empirically estimate the values of α and β.

BS are innovative because they prove that there is no need to care about α and β. As Sprenkle (1961) shows, it is impossible to accurately estimate the discount factor of a risky asset. If an asset is riskless, however, it is much easier to estimate the discount factor because there are virtually riskless financial securities, such as the U.S. treasury bills, with explicit interest rates. BS show that an instantaneously riskless portfolio can be made with a long position in stock and a short position in European call option under certain conditions, and derive the value of the option from this riskless portfolio. Because the portfolio is instantaneously riskless regardless of how risky the stock and the option are, BS use the risk-free interest rate as the discount factor in the option pricing equation.

BS assume ideal conditions as follows:

a) The short-term interest rate is known.
b) The stock price follows a geometric Brownian motion in continuous time with a constant variance rate.
c) The stock pays no dividends.
d) The option is European, that is, it can only be exercised at maturity.
e) There are no transaction costs.
f) It is possible to borrow any fraction of a security at risk-free rate.
g) There are no penalties to short selling.

Under these assumptions, BS define the stock price process as follows:

$$dS = \mu S dt + \sigma S dz \ , \tag{2.2}$$

where S is the stock price, μ is the drift rate (expected rate of return), dt is the change in time, σ is the standard deviation (volatility) of the stock price, dz is a standardized Wiener process. If the stock price follows the price process (2.2), and the price of call option f is a function of the stock price S and time t, the price process of f can be defined using Itô's Lemma as follows:

$$df = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\right) dt + \frac{\partial f}{\partial S}\sigma S dz \; . \tag{2.3}$$

Because S and f are affected by the same Wiener process dz, we can eliminate dz and construct a instantaneously riskless portfolio Π by weighting each asset as follows:

$$\Pi = -f + \frac{\partial f}{\partial S} S \; . \tag{2.4}$$

And the change in the value of portfolio Π in the time interval Δt is:

$$\Delta\Pi = -\Delta f + \frac{\partial f}{\partial S}\Delta S \; . \tag{2.5}$$

Substituting from (2.2) and (2.3) into (2.5), we find that the change in the value of the portfolio Π is as follows:

$$\Delta\Pi = \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right) \Delta t \; . \tag{2.6}$$

Since the portfolio is riskless, the rate of return of portfolio Π must be equal to the risk-free interest rate. That is:

$$\Pi = -r\Pi\Delta t \; , \tag{2.7}$$

where r is the risk-free rate. Substituting from (2.4) and (2.6) into (2.7), and rearranging, we obtain:

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S} rS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 = rf \; . \tag{2.8}$$

If $f(S,T)$ is the price of a call option for a stock S with time to maturity T, we know that:

$$f(S,T) = \max(S-K,0) \; , \tag{2.9}$$

where K is the strike price. There is only one formula $f(S,t)$ that satisfies the differential equation (2.8) subject to the boundary condition (2.9), and this formula must be the option pricing formula.

To solve this differential equation, BS make the following substitution:

$$f(S,T) = e^{rT} y \left[\frac{2\rho}{\sigma^2} \left(\ln \frac{S}{K} - \rho \right) T - \frac{2\rho^2 T}{\sigma^2} \right] , \tag{2.10}$$

where $\rho = r - \sigma^2/2$. With this substitution, (2.8) becomes:

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial S^2} , \tag{2.11}$$

and the boundary condition becomes:

$$y(u,0) = \begin{cases} 0 & \text{if } u < 0, \\ K \left[e^{\frac{u\sigma^2/2}{r-\sigma^2/2}} - 1 \right] & \text{if } u \geq 0. \end{cases} \tag{2.12}$$

BS show the solution of (2.11) as follows:

$$y(u,S) = \frac{1}{\sqrt{2\pi}} \int_{\frac{-u}{\sqrt{2S}}}^{\infty} K \left[e^{\frac{(u+q\sqrt{2S})\sigma^2/2}{r-\sigma^2/2}} - 1 \right] e^{-q^2/2} dq . \tag{2.13}$$

Substituting from (2.13) into (2.10), and simplifying, BS find:

$$f(S,t) = SN(d_1) - Ke^{-rT} N(D_2) , \tag{2.14}$$

where

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} , d_2 = \frac{\ln(S/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} .$$

This is the Black-Scholes option pricing equation.

2.2. **Counterexamples to the assumption.** BS derive an explicit closed-form solution for the price of a European call option. This solution works perfect, however, only under the 'ideal conditions' that we described above. The 'ideal conditions' are quite restrictive, and it is virtually impossible for an actual market to satisfy all of them. Some studies right after BS investigate whether some of the conditions can be released. Merton (1973) demonstrates that BS still provide the correct option price even when the interest rate is stochastic, the stock pays dividends, and the option is exercisable prior to expiration(i.e., is American). Thorp (1973) also shows that penalties against the short sales do not invalidate BS.

As Merton (1973) points out, however, there is one critical assumption in the 'ideal conditions': the assumption that the stock price follows a geometric Brownian motion in continuous time with a constant variance rate. Merton (1973) shows that the stock price must satisfy this assumption to validate the theoretical option price provided by the BS equation. However, there are a number of evidences which show that the assumption is not reliable. The stock market crash of October 1987 is a good example to show that a price is sometimes too volatile to believe that it follows a geometric Brownian motion. Before the crash, the realized annual standard deviation of stock market returns was around 20 percent. On October 19, 1987, the two month S&P futures price declined 29 percent in a day. As Jackworth and Rubinstein (1996) suggest, this is a -27 standard deviation event with probability 10^{-160}, if the stock market return follows a geometric Brownian motion. Given these facts, some studies argue that the stock price follows not only a pure diffusion process but also a Poisson jump process.

Another evidence which falsifies the assumption is so-called 'volatility smile'. Volatility smile is an anomaly that the implied volatilities of options with same underlying asset, same maturity and different strike price vary each other.[1] Since we can derive the theoretical price of an option from the volatility and other observable variables, we also can inversely extract the volatility implied by an option from the market price of an option and other variables. If the volatility rate is constant and options have same underlying asset and maturity, implied volatilities of these options should be identical regardless of their strike price. However, many studies show that this is not true. Figure 1 demonstrates a typical form of volatility smile in the S&P 500 European index options market. The figure shows that option with strike price further from the underlying asset price tends to imply higher variance. This is inconsistent with the assumption of a constant variance rate, and BS cannot explain this anomaly. Given these facts, some studies argue that the variance rate is not constant and the diffusion process does not follow a normal distribution.

The stock market crash of 1987 also affected the volatility smile and made it even harder to explain. Before the crash, the implied volatility showed a form of symmetric smile when plotted as a function of strike price. As Rubinstein (1994) shows, however, the function started to show a monotonically declining trend after the 1987 crash.[2] Figure 2 demonstrates this trend. Although most studies agree that the fear about a sudden and sharp decline of the stock prices changed the shape of the volatility smile, their theories vary in how the fear actually affects the smile. Some argue that the crash changed the expected probability distribution of the stock price in the investors' mind; others argue that the increased demand of put options with low strike prices as a portfolio insurance is a main reason.

As seen above, various ideas suggest the way to cope with the counterexamples of the price process assumption of BS. Among the ideas, the three major ones are as follows:

a) The stock price follows not only a pure diffusion process but also a Poisson jump process.
b) The variance rate is not constant and the diffusion process does not follow a normal distribution.
c) The supply and demand curves affect option price and implied volatility.

A number of theories and models are developed from these ideas, and all of them have their own outstandingness as each of them derives a way to deal with a certain problem in the price process assumption. In the next three chapters, we examine the major theories and show how they try to solve the problem.

[1] This anomaly is called 'volatility smile' because implied volatility is often smile-shaped when plotted as a function of strike price.

[2] Volatility smile is also called 'volatility skew' or 'volatility sneer' for this reason.

3. Parametric approaches

3.1. **Jump models.**

3.1.1. *Pure jump model.* Cox and Ross (1975, 1976) point out that the diffusion process is only one of two general classes of continuous time stochastic processs. Cox and Ross (1975) argue that new information tends to arrive at a market in discrete lumps rather than in a smooth flow, and assets in such markets are likely to have discontinuous jumps in value. Given these rationales, they suggest a simple form of jump process as follows:

$$dS = \mu_s dt + \sigma_s dq \ , \tag{3.1}$$

where μ_s is the drift rate function, dq is a Poisson process which takes the value 0 with probability $1 - \lambda dt$ and 1 with probability λdt. λ is the jump intensity which declares the probability of jump, and σ_s is the jump amplitude function.

As Cox and Ross (1976) show, if there is a sufficiently regular function for an option value $f(S, t)$, which is the value of option f on the underlying asset S in (3.1) at time t, the differential movement in f can be defined in a similar form:

$$df = \mu_f dt + \sigma_f dq \ , \tag{3.2}$$

where dq is the Poissson process in (3.1). The functions μ_f and σ_f depends on the unknown function f and the known values of S and t. Cox and Ross (1976) also show that if there is a riskless asset which individuals can borrow and lend freely with risk-free rate r, and the jump amplitude σ_s and σ_p are not random, it is possible to consturct a hedge portfolio with the asset S and option f such that:

$$w_S \sigma_S (dq/S) + w_f \sigma_f (dq/f) = 0 \ ,$$

or

$$w_S (\sigma_S/S) + w_f (\sigma_f/f) = 0 \ , \tag{3.3}$$

where w_S and w_f are the portfolio weights in the asset and the option respectively. In this case, risk-neutral option pricing is possible because this portfolio is riskless and there rate of return must be same as the riskless rate:

$$w_S (\mu_S/S) + w_f (\mu_f/f) = (w_s + w_f) r \ . \tag{3.4}$$

3.1.2. *Jump-diffusion model.* Merton (1976) points out that, however, it is impossible to employ this kind of risk-neutral option pricing when the underlying price process is a mixture of a diffusion process and a Poisson process as follows:

$$dS = \mu_S dt + \sigma_z dz + \sigma_q dq \ , \tag{3.5}$$

where μ_S is the drift rate function, σ_z^2 is the instantaneous variance rate function(conditional on no Poisson event), dz is a standardized Wiener process, σ_q is the jump amplitute function, dq is a Poisson process, dz and dq are independent each other. More specifically, Merton (1976) defines the process (19) as below:

$$dS = (\alpha_S - \lambda k_S)Sdt + \sigma_S S dz + (Y_S - 1)S dq \ , \tag{3.6}$$

where α is the instantaneous expected return on the asset, λ_S is the jump intensity, $k_S \equiv E[Y_S - 1]$ where $(Y_S - 1)$ is the random variable jump amplitude and E is the expectation operator, σ_S^2 is the instantaneous variance of the return(conditional on no Poisson event). Also in this case, the price process of the option $f(S,t)$ on asset S can be written in a similar form:

$$df = (\alpha_f - \lambda k_f)f dt + \sigma_f f dz + (Y_f - 1)f dq \ , \tag{3.7}$$

where dq is the Poisson process in (3.6).

Using Itô's lemma for the diffusion part and an analogous lemma for the jump part, Merton (1976) derives the following important relationships:

$$\alpha_f = \left[\frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 + \frac{\partial f}{\partial S}(\alpha - \lambda k)S + \frac{\partial f}{\partial t} + \lambda E[f(SY,t) - f(S,t)]\right] / f(S,t) \ , \tag{3.8}$$

$$\sigma_f = \frac{\partial f}{\partial S}\sigma S / f(S,t) \ , \tag{3.9}$$

where E is the expectation operator.

If a jump occurs in the asset price S and the random variable Y_S takes on the value of y, the random variable Y_f takes on the value of:

$$Y_f = f(Sy,t)/F(S,t) \ . \tag{3.10}$$

Although the variable Y_f is perfectly dependent to Y_S, their relationship is not linear because f is a non-linear function of S. Because of this non-linear relationship, it is unable to construct a riskless portfolio with S, f and a riskless asset and conduct the risk-neutral pricing. If a portfolio is constructed to perfectly hedge the diffusion risk dz, it cannot perfectly hedge the jump risk dq.

To derive the pricing equation for options which follow the mixed jump-diffusion process, Merton (1976) employs an alternative method of Ross (1976).[3] Merton (1976) assumes the conditions below:

a) The jump components of assets' return are contemporaneously independent.
b) There are n stocks outstanding.

Consider one constructs a BS hedge portfolio for each of the n stocks under these conditions. Then the value P_i of the hedge fortpolio for stock i follows the process of:

$$dP_i = (\alpha_i - \lambda_i k_i)P_i dt + P_i dq_i \ , \quad i = 1, 2, ...n. \tag{3.11}$$

[3]Although Merton (1976) also derives the same pricing equation with the Capital Asset Pricing Model (CAPM) of Sharpe (1973) and Lintner(1965), he does not strongly claim for the empirical robustness of CAPM.

If a portfolio H is again constructed with these hedge portfolios and a riskless asset where w_i is the fraction of the portfolio with stock i, the price process of portfolio H is as follows:

$$dP_H = (\alpha_H - \lambda_H k_H)P_H dt + P_H dq_H \ , \tag{3.12}$$

where

$$\alpha_H \equiv \sum_{i=1}^{n} w_i(\alpha_i - r) + r, \tag{3.13}$$

$$\lambda_H k_H \equiv \sum_{i=1}^{n} w_i \lambda_i k_i, \tag{3.14}$$

$$dq_H \equiv \sum_{i=1}^{n} w_i dq_i, \tag{3.15}$$

where r is the risk-free rate of return.

Suppose the portfolio weights w_i are restricted so that they can be written as $w_i \equiv \mu_i/n$, where the μ_i are finite constants independent of n. Ross (1976) defines the 'well-diversified' portfolio as a portfolio with n large enough. If we define s_i as $ds_i = \mu_i dq_i$, ds has an instantaneous expected value of $\mu_i \lambda_i k_i$ and an instantaneous variance rate of $\lambda_i \mu_i^2 \mathrm{Var}(Y_i - 1)$ per unit time, where $(Y_i - 1)$ is the jump amplitute of ith hedge portfolio. With (3.15), dq_H can be defined as follows:

$$dq_H = \left(\sum_{i=1}^{n} ds_i \right) /n \ , \tag{3.16}$$

where ds_i are independent each other. Therefore, $dq_H \to \lambda_H k_H dt$ with probability one as $m \to \infty$. In other words, the portfolio H becomes virtually riskless as the number of smaller hedge portfolios in H becomes large and H gets well-diversified. Thus, the rate of return dH/H will be its expected return $\alpha_H dt$ with probability one, and this rate should be the risk-free rate r to prevent arbitrage. Substituting this condition into (3.13), we have that for large m,

$$\frac{1}{n} \sum_{i=1}^{n} \mu_i(\alpha_i - r) = 0 \ . \tag{3.17}$$

Because u_i are arbitrary, (3.17) must hold for all proper choices for u_i. Thus,

$$\alpha_i = r, \quad \text{for } j = 1, 2, ..., n \ . \tag{3.18}$$

Merton (1976) shows that if $\alpha_i = r$,

$$(\alpha_S - r)/\sigma_S = (\alpha_f - r)/\sigma_f \ . \tag{3.19}$$

With (3.8), (3.9) and (3.19), an option price f must satisfy:

$$0 = \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 + \frac{\partial f}{\partial S}(r - \lambda k)S - \frac{\partial f}{\partial t} - rf + \lambda E[f(SY, \tau) - f(S, \tau)], \tag{3.20}$$

under the boundary condition of

$$f(0, \tau) = 0, \quad f(S, 0) = \max[0, S - K], \tag{3.21}$$

where τ is the time at maturity, and K is the strike price of the option. As in the BS equation, (3.20) does not depend on the expected rates of return. Instead, only the risk-free rate r appears. If there are no jumps and therefore $\lambda = 0$, (3.20) becomes (2.8).

To derive a solution to (3.20),[4] Merton (1976) uses the BS option price $f_{BS}(S, T; K, r, \sigma^2)$, which can be defined by (2.14). If we define a random variable X_n to have the same distribution as the product of n i.i.d. random variables, each identically distributed to the Y_S in (3.6), and $X_0 \equiv 1$, the solution to (3.20) for $f(S, \tau)$ can be defined as follows:

$$f(S, \tau) = \sum_{n=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!}(E[f_{BS}(SX_n e^{-\lambda k\tau}, \tau; K, \sigma^2, r)]) \ , \tag{3.22}$$

where E is the expectation operator.

A critical shortcoming in (3.20) is that X_n might be too complicated. To deal with this problem, an additional assumption can be employed to simplify the formula. If we assume that, for instance, the random variable Y_S has a log-normal distribution, (3.20) can be rewritten as follows:

$$f(S, \tau) = \sum_{n=0}^{\infty} \frac{e^{-\lambda'\tau}(\lambda'\tau)^n}{n!} f_{BS}(S, \tau; K, v_n^2, r_n) \ , \tag{3.23}$$

where $\lambda' = \lambda(1 + k)$, $v_n^2 = \sigma^2 + n\delta^2/\tau$, $\delta^2 = \mathrm{Var}(\log Y)$, $r_n = r - \lambda k + n\gamma/\tau$, $\gamma \equiv \log(1 + k)$.

3.2. **Stochastic volatility models.** In the empirical literature, many studies (e.g., Clark, 1973; and Blattberg and Gonedes, 1974) support the idea that the volatility of stock price changes over time. Based on these evidences, various types of stochastic volatility (SV hereafter) model is developed to explain the changing volatility. A SV model can be defined as a model in where the volatility follows a Wiener process which is different from that of the price process. Hull and White

[4]Merton (1976) mentions that this solution is not a complete closed-form, but is a partial solution which is in a reasonable form for computation.

(1987) show a general form of SV model as follows:

$$dS = \mu S dt + \sigma S dz_1 \ , \quad dV = fV dt + \zeta V dz_2 \ , \tag{3.24}$$

where $V = \sigma^2$, and dz_1 and dz_2 are two different Wiener processes with the correlation coefficient ρ.

A SV model can be more specified by assuming a non-zero ρ or some dependence among the variables, or setting some of the variables as functions. To properly specify a model, the assumptions must be realistic and be supported by empirical evidences. There are many versions of SV models each of which tries to explain some empirical phenomena with different parameters.

3.2.1. *Constant elasticity of variance (CEV) models.* Black (1975) points out that a stock that drops in price is likely to show a higher volatility in the future than a stock that rises in price. Black (1975) and Schmalensee and Trippi (1978) attribute this inverse relationship to the leverage effect. According to them, if the stock price of a firm falls the stock becomes risky in two ways. First, the financial leverage increases because the market value of the firm's equity tends to fall more rapidly than the market value of the firm's debt when stock price falls. Second, the operation leverage increases because the income of the firm tends to fall more rapidly than the cost of the firm because some of the cost is fixed. Beckers (1980) supports this idea by showing that the financial leverage is negatively related to the stock return.

Based on this inverse relationship, Cox (1975, as cited in Beckers, 1980) assumes the variance of the price as a decreasing function of the price itself. According to him, the price process dS in (3.24) becomes:

$$dS = \mu S dt + \delta S^{(\theta/2)} dz_1 \ , \tag{3.25}$$

where $\delta > 0$ and $0 \leq \theta < 2$ are constants. This means that:

$$f = \frac{dG}{V dt} \ , \quad \zeta = 0 \ , \tag{3.26}$$

where $G = \delta^2 S^{(\theta-2)}$. This price process is called the CEV diffusion process. Strictly speaking, CEV process is not a SV process because volatility is a deterministic function here. However, we should know about this process and its rationale to understand the Heston (1993)'s model in 3.2.3.

Cox and Ross (1976) introduce a risk-neutral option pricing formula for a call option $C(S, \tau)$ whose underlying follows the CEV diffusion:

$$C(S,\tau) = (S - Ke^{-r\tau})N(y_1) + (S + Ke^{-r\tau})N(y_2) + v(n(y_1) - n(y_2)) \ , \tag{3.27}$$

where

$$v = \sigma \left(\frac{1 - e^{-2r\tau}}{2r} \right)^{1/2} \ , \quad y_1 = \frac{S - Ke^{-r\tau}}{v} \ , \quad y_2 = y_1 - \frac{2S}{v} .$$

In this expression, $N(x)$ is the cumulative standard normal density function, and $n(x)$ is the standard normal density function.

3.2.2. *Mean-reverting models.* Many studies (e.g., Beckers, 1983; Poterba and Summers, 1986; Fama and French, 1988) show that shocks to volatility decay over time and therefore the volatility tends to revert to its long-term mean. To capture this phenomenon, Scott (1987) introduces a Ornstein-Uhlenbeck process into the volatility process as follows:

$$dS = \mu S dt + \sigma S dz_1 \ , \quad d\sigma = \kappa(\theta - \sigma)dt + \zeta\sigma dz_2 \ , \tag{3.28}$$

where κ is the speed of adjustment to the long-run mean θ.

If an asset follows this process, two series of options on this asset with differnt maturities are needed to construct a riskless portfolio with the asset. By constructing this portfolio, Scott (1987) derives the partial differential equation for an option price $f(S, \sigma, \tau)$:

$$\begin{aligned} -\frac{\partial f}{\partial \tau} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 + \frac{\partial^2 f}{\partial S \partial \sigma}\rho\zeta\sigma S + \frac{1}{2}\frac{\partial^2 f}{\partial \sigma^2}\zeta^2 - rf + \frac{\partial f}{\partial S}rS \\ + \frac{\partial f}{\partial \sigma}[\kappa(\theta - \sigma) - \lambda] = 0 \ , \end{aligned} \tag{3.29}$$

where λ is the risk premium associated with $d\sigma$.

Although the assumption of mean-reverting is quite realistic, this paper also has some limits. First, this paper ignores the presence of risk premium by setting $\lambda = 0$. Second, this paper cannot derive a closed-form solution for the option price. Scott (1987) suggests the Monte Carlo simulations method to estimate option prices. Wiggins (1987) shows that the risk-neutral pricing is available for an option on the market, but he also fails to derive a closed-form solution for the option prices on individual stocks. Given this problem, most of the SV models before Heston (1993) use numerical methods to estimate parameters.

3.2.3. *Heston (1993)'s model.* Heston (1993) points out that the existing SV models fail to provide a closed-form solution(e.g., Scott, 1987; Wiggins, 1987; Hull and White, 1987), or to capture the inverse relationship between asset price and volatility (e.g., Jarrow and Eisenberg, 1991; Stein and Stein, 1991). To fill the gap, he derives a model which provides a closed-form solution for a oprion price whose underlying asset follows a SV process, while including all the concepts of mean-reverting volatility, inverse relationship between stock price and volatility, and non-zero volatility risk premium.

Heston (1993) assumes that the variance of an asset price follows a square-root process as follows:

$$dS = \mu S dt + \sqrt{V} S dz_1 \ , \quad dV = \kappa(\theta - V)dt + \zeta\sqrt{V} dz_2 \ . \tag{3.30}$$

In this model, the partial differential equation for a call option $f(S, V, t)$ is derived as follows:

$$\begin{aligned}\frac{\partial f}{\partial t}+\frac{1}{2}\frac{\partial^2 f}{\partial S^2}V^2S^2+\frac{\partial^2 f}{\partial S\partial\sigma}\rho\zeta\sigma S+\frac{1}{2}\frac{\partial^2 f}{\partial\sigma^2}\zeta^2V-rf+\frac{\partial f}{\partial S}rS\\ +\frac{\partial f}{\partial V}[\kappa(\theta-\sigma)-\lambda(S,V,t)]=0\ .\end{aligned}\tag{3.31}$$

To estimate the price of volatility risk λ, he employs the methodology of Breeden (1979):

$$\lambda(S,V,t)=\gamma\mathrm{Cov}[dV,dC/C]\ ,\tag{3.32}$$

where C is the consumption rate and γ is the relative-risk aversion of an investor. Cox et al. (1985) derive the consumption process dC as follows:

$$dC=\mu_C VCdt+\sigma_C\sqrt{V}Cdz_3\ ,\tag{3.33}$$

where dz_3 is another Wiener process and dC has a constant correlation with dS. Given (3.32) and (3.33), the risk premium is proportional to λV. Heston (1993) points out that, in theory, one can extract λ from a volatility dependent asset in the same way which is used to extract implied volatility.

In this model, a European call option with strike price K and maturing at time T satisfies (3.31) under the following boundary conditions:

$$\begin{aligned}f(S,V,T)=\max(0,S-K),\\ f(0,V,t)=0,\\ \frac{\partial f}{\partial S}(\infty,V,t)=1,\\ rS\frac{\partial f}{\partial S}(S,0,t)+\kappa\theta\frac{\partial f}{\partial V}(S,0,t)-rU(S,0,t)+\frac{\partial f}{\partial t}(S,0,t)=0,\\ f(S,\infty,t)=S.\end{aligned}\tag{3.34}$$

Based on (3.31) and (3.34), Heston (1993) writes the option pricing equation as follows:

$$f(S,V,t)=SP_1-Ke^{-r(T-t)}P_2\ .\tag{3.35}$$

If $x=\ln S$, P_1 and P_2 must satisfy the following equation:

$$\begin{aligned}\frac{1}{2}\frac{\partial^2 P_j}{\partial x^2}V+\frac{\partial^2 P_j}{\partial x\partial V}\rho\zeta V+\frac{1}{2}\frac{\partial^2 P_j}{\partial V^2}\zeta^2V+\frac{\partial P_j}{\partial x}(r+u_jV)\\ +\frac{\partial P_j}{\partial V}(a_j-b_jV)+\frac{\partial P_j}{\partial t}=0\ ,\end{aligned}\tag{3.36}$$

for $j=1,2$, where $u_1=1/2$, $u_2=-1/2$, $a=\kappa\theta$, $b_1=\kappa+\lambda-\rho\zeta$, $b_2=\kappa+\lambda$.

(3.36) are subject to the following terminal condition:

$$P_j(x, V, T; \ln K) = 1_{\{x \geq \ln K\}} \ . \tag{3.37}$$

Given (3.30), x follows the process of:

$$dx = (r + u_j V)dt + \sqrt{V} dz_1 \ , \quad dV = (a_j - b_j V)dt + \zeta\sqrt{V} dz_2 \ , \tag{3.38}$$

where u_j, a_j and b_j are those in (3.36).

Using a Fourier transformation, Heston (1993) proves that:

$$P_j(x, V, T; \ln K) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi \ln K} f_j(x, V, T; \phi)}{i\phi}\right] d\phi \ , \tag{3.39}$$

for $j = 1, 2$, where $i = \sqrt{-1}$. $f_1(x, V, T; \phi)$ and $f_2(x, V, T; \phi)$, the caracteristic function of P_j, can be solved as follows:

$$f_j(x, V, T; \phi) = e^{C(T-t;\phi) + D(T-t;\phi)V + i\phi x}, \tag{3.40}$$

where

$$C(\tau, \phi) = r\phi i\tau + \frac{a}{\sigma^2}\left\{(b_j - \rho\zeta\phi i + d)\tau - 2\ln\left[\frac{1 - ge^{d\tau}}{1 - g}\right]\right\},$$

$$D(\tau, \phi) = \frac{b_j - \rho\zeta\phi i + d}{\zeta^2}\left[\frac{1 - e^{d\tau}}{1 - ge^{d\tau}}\right],$$

$$g = \frac{b_j - \rho\zeta\phi i + d}{b_j - \rho\zeta\phi i - d}, \qquad d = \sqrt{(\rho\zeta\phi i - b_j)^2 - \zeta^2(2u_j\phi i - \phi^2)}.$$

$f_1(x, V, T; \phi)$ and $f_2(x, V, T; \phi)$ also satisfy the equation (3.36), subject to the terminal condition of:

$$f_j(x, V, T; \phi) = e^{i\phi x} \ . \tag{3.41}$$

(3.35), (3.39) and (3.40) give the closed-form solution for a European call option price $f(S, V, t)$.

3.3. **GARCH option pricing models.** As shown above, jump models and stochastic volatility models deal with heteroskedasticity by introducing an additional source of uncertainty, such as a Poisson, Wiener or Ornstein-Uhlenbeck process. In this way, a model can explain heteroskedasticity by fluctuating volatility with the new process. On the other hand, some option pricing models try to explain heteroscedasticity not by the additional source of uncertainty but by the lagged effect of market shocks. The Generalized Autoregressive Conditional Heteroskedasticity (GARCH) process and its descendants provide a good alternative for those models.

The GARCH process, which is first proposed by Bollerslev (1986), is a generalized version of the Autoregressive Conditional Heteroskedasticity (ARCH) process of Engle (1982). If an one-period asset return y follows a simple ARCH(p) process, y at time t can be defined as follows:

$$y_t = r + \epsilon_t, \quad \epsilon_t \mid \phi_{t-1} \sim N(0, h_t), \quad h_t = \alpha_0 + \sum_{i=1}^{p} \alpha_i \epsilon_{t-i}^2, \tag{3.42}$$

where r_t is the risk-free rate of return at time t, ϕ_t is the information set available at time t, dz is a standard Wiener process. In (3.42), we can call h_t the conditional variance of y_t.

When an ARCH(p) process is assumed, $p+1$ parameters need to be estimated. Therefore, there might be some loss in accuracy if q is large. To deal with this problem, Bollerslev (1986) proposes a GARCH(p,q) process as follows:

$$h_t = \alpha_0 + \sum_{i=1}^{p} \alpha_i \epsilon_{t-i}^2 + \sum_{j=1}^{q} \beta_j h_{t-j} \ , \tag{3.43}$$

where y_t and ϵ_t are same as those in (3.42). In a GARCH(p,q) process, the lagged h terms capture the geometric lag structure of e^2 terms. The GARCH(1,1) process is a very popular specification because it explains many time series data well.

3.3.1. *GARCH-M option pricing model.* Duan (1995) first introduces the GARCH process into option pricing. He chooses the GARCH-in-Mean (GARCH-M) process to consider the effect of risk premium. Engle and Bollerslev (1986) propose the GARCH-M process to explain CAPM with heteroscedasticity. Under CAPM, the expected rate of return becomes larger as the invested asset becomes more risky, i.e., more volatile. To consider this relationship, an additional volatility term can be added to the drift part of GARCH process:

$$y_t = r + \lambda\sqrt{h_t} + \epsilon_t \ , \tag{3.44}$$

where λ is the unit risk premium, ϵ_t and h_t are same as those in (3.43). Duan (1995) takes the GARCH-M(p,q) process for the log-return of asset S in the following form:

$$\ln \frac{S_t}{S_{t-1}} = r + \lambda\sqrt{h_t} - \frac{1}{2}h_t + \epsilon_t \ . \tag{3.45}$$

In this way, one plus the conditionally expected rate of return can be set to $\exp(r + \lambda\sqrt{h_t})$ under conditional lognormality.

When an option pricing model assumes GARCH process for the asset return, the model cannot apply the risk-neutral framework of BS because GARCH is a discrete-time process which defines the conditional asset movement. To deal with this problem, Duan (1995) introduces a generalized version of risk-neutralization which is called the Locally Risk-neutral Valuation Relationship (LRNVR). A pricing measure Q satisfies LRNVR if:

a) Q is mutually absolutely continuous with respect to measure P.
b) $S_t/S_{t-1} \mid \phi_{t-1}$ follows a lognormal distribution under Q.
c) $\mathrm{Var}^Q(\ln(S_t/S_{t-1}) \mid \phi_{t-1}) = \mathrm{Var}^P(\ln(S_t/S_{t-1}) \mid \phi_{t-1})$.

Duan (1995) shows that LRNVR holds under some condition. When a measure Q satisfies LRNVR, the GARCH(p,q) process transforms into the following form under measure Q:

$$\ln \frac{S_t}{S_{t-1}} = r - \frac{1}{2} h_t + \epsilon_t, \tag{3.46}$$

where

$$\epsilon_t \mid \phi_{t-1} \sim N(0, h_t),$$
$$h_t = \alpha_0 + \sum_{i=1}^{p} \beta_i h_{t-i} + \sum_{j=1}^{q} \alpha_j (\epsilon_{t-j} - \lambda \sqrt{h_{t-j}})^2.$$

Under this process, the discounted asset price process $e^{-rt} S_t$ is a Q-martingale. Therefore, the value of a European call option C at time t can be defined as follows:

$$C_t = e^{-(T-t)} E^Q [\max(S_T - K, 0) \mid \phi_t] \,, \tag{3.47}$$

where T is the maturity and K is the strike price of C. Duan (1995) also points out that the analytic solution is not available for (3.47), and that a Monte Carlo simulation method should be used to compute the option price.

3.3.2. *Closed-form GARCH option pricing model.* As seen in the case of Duan (1995), GARCH option pricing model can value an option solely on the observable variables because the volatility is clearly observable in the GARCH framework. This is a strong advantage of GARCH models over SV models which cannot compute the exact current volatility from the historical data. Although SV models can use the implied volatility as an alternative, this is also not always possible because option prices are not reliable in illiquid markets.

However, GARCH option models in the early stage also have a shortcoming that they do not provide a closed-form solution for option price. Engle and Mustafa (1992), Amin and Ng (1993), as well as Duan (1995) try to solve the option pricing equation by the Monte Carlo simulation method. In contrast, Heston and Nandi (2000, HN hereafter) provides a closed-form solution for option price in the GARCH framework. Moreover, this model also introduces the concept of risk premium and the negative correlation between stock price and volatility.

HN assume that the underlying asset S follows a GARCH process as follows:

$$\ln \frac{S_t}{S_{t-1}} = r + \lambda h_t + \epsilon_t, \tag{3.48}$$

where

$$\epsilon_t \mid \phi_{t-1} \sim N(0, h_t),$$
$$h_t = \alpha_0 + \sum_{i=1}^{p} \beta_i h_{t-i} + \sum_{j=1}^{q} \alpha_j \left(\frac{\epsilon_{t-j}}{\sqrt{h_{t-j}}} - \gamma_i \sqrt{h_{t-j}} \right)^2 .$$

If the process is first-order ($p = q = 1$), and $\beta_1 + \alpha_1 \gamma_1^2 < 1$ so that the process is stationary with finite mean and variance, the future conditional variance h_{t+1} can be derived as follows:

$$h_{t+1} = \alpha_0 + \beta_1 h_t + \alpha_1 \frac{(\ln(S_t/S_{t-1}) - r - \lambda h_t - \gamma_1 h_t)^2}{h_t}. \tag{3.49}$$

In (3.49), γ_1 determines the skewness and α_1 determines the kurtosis of the probability distribution. HN show that the covariance between the conditional variance and return is:

$$\mathrm{Cov}_{t-1}[h_{t+1}, \ln S_t] = 2\alpha_1\gamma_1 h(t). \tag{3.50}$$

If both α_1 and γ_1 are positive, there is a negative correlation between the conditional variance and return.

Foster and Nelson (1994) show that the process of h_t converges to the mean-reverting square-root process of dV in (3.30) as the gap between discretized times converges to zero. Given this fact, HN argue that the process (3.48) contains the SV model of Heston (1993) as a special case so that it is possible to derive a closed-form solution for option price. To know the risk-neutral probability distribution of S_t, HN rearrange (3.48) as follows:

$$\ln \frac{S_t}{S_{t-1}} = r - \frac{1}{2} h_t + \epsilon_t^*, \tag{3.51}$$

where

$$\begin{aligned} \epsilon_t^* &= \epsilon_t + \left(\lambda + \frac{1}{2}\right) h_t, \\ h_t &= \alpha_0 + \sum_{i=1}^{p} \beta_i h_{t-i} + \sum_{j=2}^{q} \alpha_j (\epsilon_{t-j} - \gamma_i \sqrt{h_{t-j}})^2 \\ &\quad + \alpha_1 \left(\left(\frac{\epsilon_{t-1}^*}{\sqrt{h_{t-1}}} \right) - \gamma_1^* \sqrt{h_{t-1}} \right)^2, \\ \gamma_1^* &= \gamma_1 + \lambda + \frac{1}{2}. \end{aligned}$$

Because there is no reason for the risk-neutral distribution of ϵ_t^* to be normal, HN additionally assume that the value of a call option with one period to expiration obeys the Rubinstein's (1984) formula. Given this assumption, (3.51) becomes the risk-neutral version of (3.48).

To derive a closed-form solution for option price, HN define a conditional generating function $f(\phi)$:

$$f(\phi) = E_t[S_t^{\phi}]. \tag{3.52}$$

Under the process (3.48), $f(\phi)$ can be defined as follows:

$$
\begin{aligned}
f(\phi) = S_t^{\phi}\exp\Bigg(&A(t;T,\phi) + \sum_{i=1}^{p} B_i(t;T,\phi)h_{t+2-i} \\
&+\sum_{j=1}^{q-1} C_i(t;T,\phi)\Big(\frac{\epsilon_{t+1-j}}{\sqrt{h_{t+1-j}}} - \gamma_j\sqrt{h_{t+1-j}}\Big)^2\Bigg),
\end{aligned}
\tag{3.53}
$$

where

$$
\begin{aligned}
A(t;T,\phi) &= A(t+1;T,\phi) + \phi r + B_1(t+1;T,\phi)\alpha_0 \\
&\quad -\frac{1}{2}\ln(1-2\alpha_1 B_1(t+1;T,\phi)), \\
B_1(t;T,\phi) &= \phi(\lambda+\gamma_1) - \frac{1}{2}\gamma_1^2 + \beta_1 B_1(t+1;T,\phi) \\
&\quad +\frac{1/2(\phi-\gamma_1)^2}{1-2\alpha_1 B_1(t+1;T,\phi)}.
\end{aligned}
\tag{3.54}
$$

HN show that, if $p = q = 1$, these coefficients can be calculated recursively from the following terminal conditions:

$$
A(T;T,\phi) = 0, \quad B(T;T,\phi) = 0. \tag{3.55}
$$

Because $f(\phi)$ is the moment generating function of the logarithm of the underlying asset price, $f(i\phi)$ is the characteristic function of the log asset price. Given this fact, HN show that:

$$
\begin{aligned}
E_t[\mathrm{Max}(S_T - K, 0)] = f(1)&\left(\frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \mathrm{Re}\left[\frac{K^{-i\phi}f(i\phi+1)}{i\phi f(1)}\right]d\phi\right) \\
&-K\left(\frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \mathrm{Re}\left[\frac{K^{-i\phi}f(i\phi)}{i\phi}\right]d\phi\right),
\end{aligned}
\tag{3.56}
$$

where Re is the real part of a complex number. Given (3.56), the value of a European call option C with strike price K can be defined as follows:

$$
\begin{aligned}
C &= e^{-r(T-t)}E_t^*[\mathrm{Max}(S_T - K, 0)] \\
&= \frac{1}{2}S_t + \frac{e^{-r(T-t)}}{\pi}\int_0^{\infty} \mathrm{Re}\left[\frac{K^{-i\phi}f^*(i\phi+1)}{i\phi}\right]d\phi \\
&\quad -Ke^{-r(T-t)}\left(\frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \mathrm{Re}\left[\frac{K^{-i\phi}f^*(i\phi)}{i\phi}\right]d\phi\right),
\end{aligned}
\tag{3.57}
$$

where E^* is the expectation operator under the risk-neutral distribution, and $f^*(i\phi)$ is the characteristic function of the log asset price under the risk-neutral distribution.

3.4. **Parameter estimation.** One of the problems the parametric approaches encounter is that they cannot observe most of their parameters in the market. Although some of the parameters can be estimated from the historical data, this way of estimation is neither practical or convenient, as Bakshi et al. (1997) point out, because of its stringent requirement on the data. To deal with this difficulty, most studies employ the implied volatility based on their models. Whaley (1982) determines the estimate of implied volatility by minimizing the sum of squared deviations between theoretical prices and the market prices. A huge number of studies (e.g., Bodurtha and Courtadon, 1987; Melino and Turnbull, 1990, 1995; Bates, 1991, 2000), follow the methodology and use the least squares approximation.

Bakshi et al. (1997) well explain the methodology. First, collect N option prices on the same stock at the same time t, where N is equal to or bigger than the number of parameters plus one. Next, set the theoretical price of each option as a function of strike price K, time to maturity $\tau$and other unknown parameters. Then, the differences between each theoretical price and market price will become a function of unknown parameters. That is, for each option i, define ϵ as:

$$\epsilon_i(\phi) \equiv \hat{f}_i(t, \tau_i; K_i) - f_i(t, \tau_i; K_i) \ , \tag{3.58}$$

where ϕ is the parameter vector, $\hat{f}_i$ is the theoretical price of option i with maturity τ and strike price K at time t, and f_i is the market price of option i. Finally, find ϕ to solve:

$$SSE(t) \equiv \min_{\phi} \sum_{i=1}^{N} |\epsilon(\phi)|^2 \ . \tag{3.59}$$

With this procedure, parameter estimates can be derived for time t.

4. Non-parametric approaches

As Cox and Ross (1976) explain, we can value an option if we know the probability distribution of the price process of the underlying asset. Various models above try to obtain the asset price dynamics in a closed form, therefore, to grasp the shape of probability distribution and calculate the optimal price. However, this way of valuation has some deficiencies. First, the closed-form solution is dependent on strong assumptions on the price process. If the assumption is invalid, therefore, the solution has no meaning. Second, the closed-form solution is unable to be derived if the assumed process is too complicated. In this situation, the probability distribution have to be estimated by numerical methods. To bypass these problems, the probability distribution can be estimated nonparametrically. When a nonparametric method is employed, there is no restriction regarding the underlying price process. A nonparametric approach is preferable, therefore, especially when it is impossible to define the price process. Considering the empirical evidences which show that the price processes are hard to be defined clearly, a nonparametric method can be an effective alternative.

The price c of a European call option with strike price K and maturity τ can be defined as follows:

$$c = e^{-r\tau} \int_{S_\tau = K}^{\infty} (S_\tau - K) f(S_\tau) dS\tau \ , \tag{4.1}$$

where r is the risk-free interest rate, $S\tau$ is the underlying asset price at maturity, and f is the risk-neutral probability density function of $S\tau$. By differentiating this pricing function twice with respect to the exercise price K, Breeden and Litzenberger (1978) show:

$$f(S_\tau) = e^{rt} \frac{\partial^2 c}{\partial K^2} \, . \tag{4.2}$$

Based on this finding, Aït-Sahalia and Lo (1998) derives the probabitlity distribution nonparametrically by estimating an option pricing formula $\hat{c}(\cdot)$ and differentiating the estimator twice with respect to K. As $\hat{c}(\cdot)$ converges to the true option pricing formula $c(\cdot)$, $\partial^2 \hat{c}(\cdot)/\partial K^2$ will converge to $\partial^2 c(\cdot)/\partial K^2$. To estimate $\hat{c}(\cdot)$, they use a set of historical option prices $\{c_i\}$ and accompanying characteristics $\{Z_i\}$, instead of parameters. $\hat{c}(\cdot)$ can be estimated by setting it come as close to $\{c_i\}$ as possible:

$$SSE(t) \equiv \min_{c(\cdot)\in\Gamma} \sum_{i=1}^{N} |c_i - c(Z_i)|^2 \, , \tag{4.3}$$

where Γ is the space of twice-continuously-differentiable functions.

5. Empirical Comparison

As we have seen above, various alternative models are devised to fill the gap between BS and the market. Each model tries to capture some characteristics of price movement in the markets and derive the optimal price more accurately. In fact, most of the models succeed in outperforming the BS model. When the models are used for in-sample and out-of-sample forecasts, they show better performance than that of BS. Given these facts, we can say that BS can be enhanced by introducing some additional components. However, a model cannot prove its value only by showing its excellence over the BS model if there are many models which show the same overperformance. Given the BS model as the worst one in terms of performance, the other models should compete each other to show their value as an alternative. Therefore, researchers conducted empirical comparisons to examine the relative performance of the alternative models and find which one is the best.

Bakshi et al. (1997) compare the pricing and hedging performance of BS, SV, SV-jump (SVJ), and SV-stochastic interest rate (SVSI) models. After an empirical analysis, Bakshi et al. (1997) argue that taking stochastic volatility into account is of the first-order importance in improving upon the BS formula. Moreover, they also show that adding the jump component to the SV model can further improve its performance, especially in pricing short-term options, whereas modeling stochastic interest rates can enhance the fit of long-term options. For hedging purposes, however, they cannot find any evidence for that incorporating either the jump or the SI feature does not seem to improve the SV model's performance further.

Kim and Kim (2003) investigates the improvement in the pricing of Korean KOSPI 200 index options when models other than BS are used. They compare empirical performances of Heston's (1993) SV model, Heston and Nandi's (2000) closed-form GARCH model, Madan et al.'s (1998) variance gamma model, and the ad hoc BS procedure in Dumas et al. (1998). They show that Heston's (1993) model outperforms the other models in terms of effectiveness for in-sample pricing, out-of-sample pricing and hedging. Also, it is shown that the stochastic volatility models cannot mitigate the volatility smiles effects found in cross-sectional options data, but can reduce the effects better than the Black and Scholes model.

An and Suo (2009) compare the performance of BS model, Merton's (1976) jump-diffusion model, Bakshi et al.'s (1997) SVJ model, and the ad hoc BS procedure in hedging exotic options such as barrier options and compound options. They demonstrate that the performance of the alternative models relative to the BS model depends on how exotic the hedged option is. The relative performance of a model in hedging barrier options is quite different from the performance of same model in hedging in compound options. SV model outperforms the BS model in hedging most types of up-and-out call options, while the jump-diffusion model and SVJ model generally demonstrate poorer performance in hedging barrier options.

6. Net buying pressure theory

So far, we have been talking about the asset price processes and ways to estimate them. As Merton (1973) points out, the assumption of pure diffusion process with a constant variance rate can be a flaw to the BS equation, and the way to release this assumption is a popular topic among the academics and practitioners. However, Bollen and Whaley (2004) suggest a totally different view of option pricing. They argue that volatility smile is not due to the price process or the probability distribution, but caused by the supply and demand function in the market. In other words, they suggest that the another assumption of BS about the efficient market—without any transaction cost—also has a problem.

According to Bollen and Whaley (2004), implied volatilities are distracted by the limits to arbitrage. After market participants experienced a sudden and steep market decline in 1987, the demand for put options on market indices as a portfolio insurance has increased.[5] Because there is not enough suppliers who are willing to take short positions, market makers have to step in to absorb the demand.[6] Market makers also have limits in their liquidity and needs to hedge their position, and they require compensation for taking their position. As a result, implied volatility will exceed actual return volatility with the difference being the compensation for market makers.

On the other hand, Kang and Park (2008) show that the limited arbitrage is not the only reason for the net buying pressure. Kang and Park (2008) investigate the KOSPI200 index options market, which has no market maker, and conclude that traders with information about future price movements raise buying and selling pressures. These show that market can become inefficient for various reasons, and market efficiency should be considered when pricing options.

7. Conclusion

Since the pioneering work of Bachelier (1900), many academics and practitioners have been trying to solve the problem of option pricing. As BS mention, an option pricing formula is also applicable to various securities such as stocks, bonds and warrants because almost all corporate liabilities can be explained as a combination of options. Although options are very abstract and ambiguous securities, it will be a

[5]The value of an index put option increases when the index declines. This is why a put option on a major market index can be regarded as an insurance.

[6]the New York Stock Exchange(NYSE), American Stock Exchange(AMEX) and London Stock Exchange(LSE) have official market makers.

great help for understanding the characteristics of financial securities if the various option pricing methods above can be approached in a comprehensive manner.

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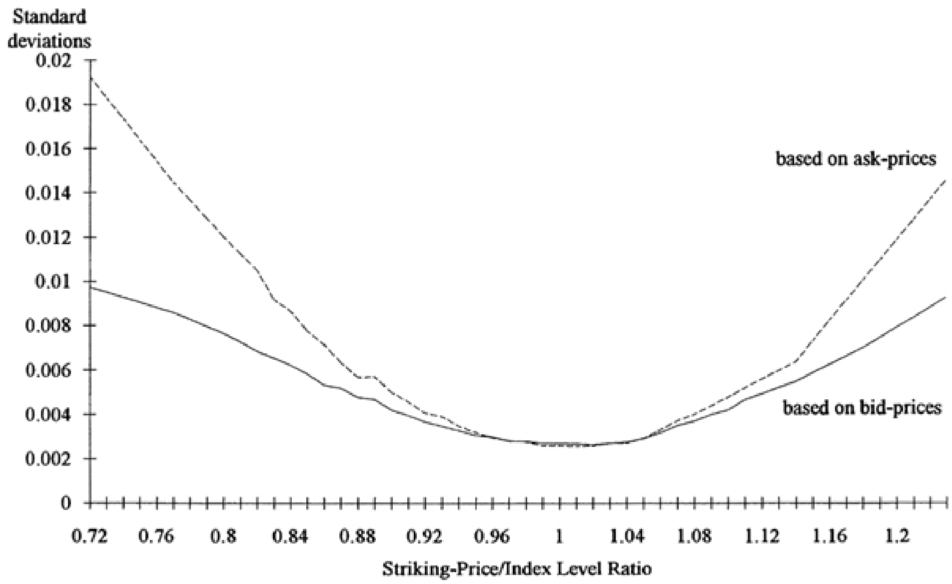

FIGURE 1. A classic form of volatility smile, in Jackwerth and Rubinstein(1996). This figure depicts the median daily standard deviation of implied volatilities grouped by the ratio of strike price to index. Samples are the S&P 500 index options data from April 2, 1986 to December 31, 1993, with time to maturity from 135 to 225 days.

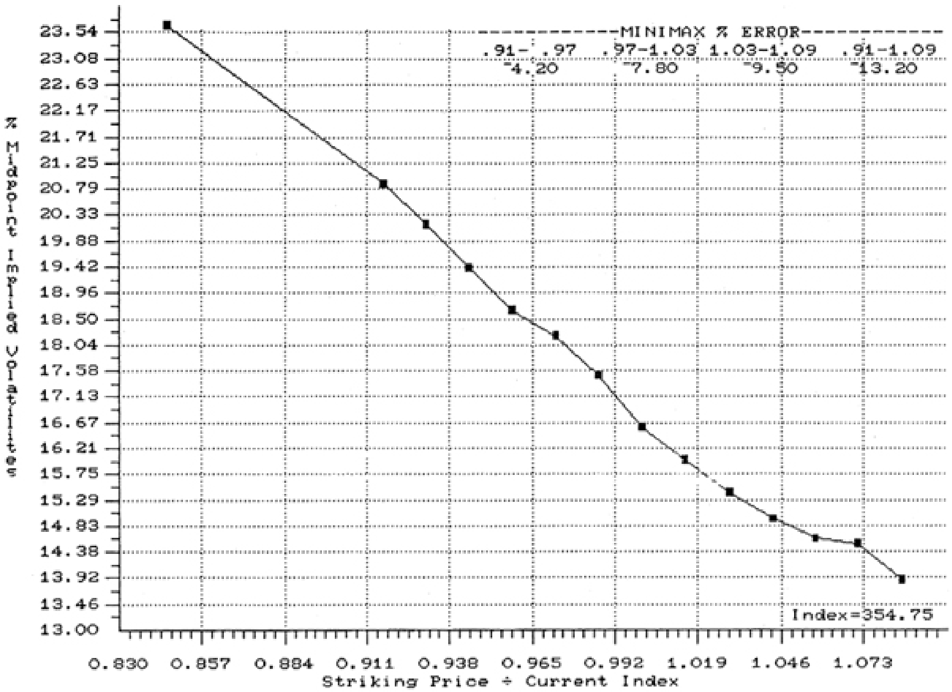

FIGURE 2. Volatility smile after the stock market crash of 1987 , in Rubinstein(1994). This figure demonstrates the monotonically declining trend in the volatility smile after the stock market crash of 1987. Samples are the S&P 500 index options data at 10:00 A.M., January 2, 1990.